Tree ideals and Cohen reals

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Trees and tree ideals

Tree type

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- $\mathbb{T}$ consists of perfect trees

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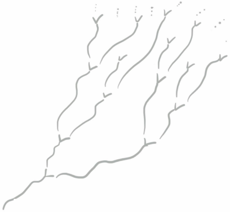
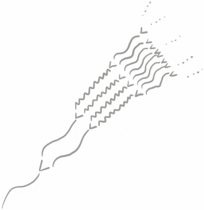
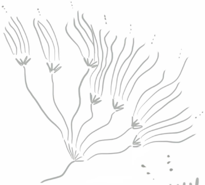
Tree type

Let $\mathbb{T}$ be a collection of trees on $A^{<\omega}$ such that:

- $\mathbb{T}$ consists of perfect trees
- $\forall T \in \mathbb{T} \; \forall \sigma \in T, \; T|_\sigma \in \mathbb{T}$
- $\forall T \in \mathbb{T} \; \exists \{S_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathbb{T}$ all below T and that $[S_\alpha] \cap [S_\beta] = \varnothing$ for $\alpha \neq \beta$ (large antichains)


The Marczewski-style tree ideal

Tree ideal

The Marczewski tree ideal t_0 consists of countable unions of $\mathbb{T}$-nwd sets, where $X \in \mathbb{T}$-nwd if

$$\forall T \in \mathbb{T}\ \exists S \leq T, S \in \mathbb{T},\ X \cap [S] = \varnothing$$

Well investigated examples:

- s_0 - Marczewski ideal
- m_0, l_0 - Miller and Laver ideal
- v_0 - Silver, Mycielski ideal

Known for not having borel basis. Even $\mathfrak{c} < cof(t_0)$

The Borel part of t_0

As t_0 does not have borel basis

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$t_0|_{\mathcal{B}or}$ = borel sets from t_0

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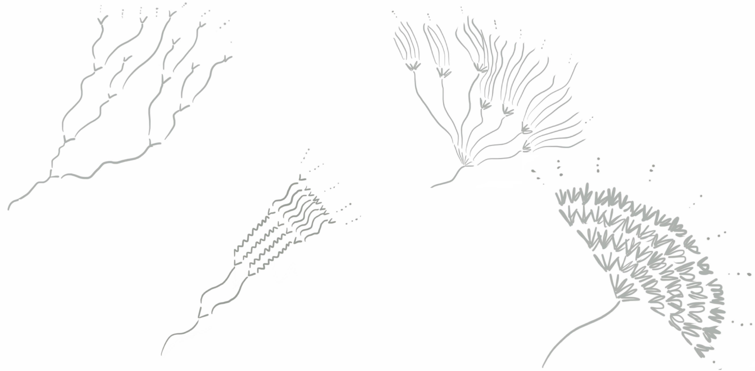
$t_0|_{\mathcal{B}or}$ = borel sets from t_0

- $s_0|_{\mathcal{B}or}$ = countable sets
- $m_0|_{\mathcal{B}or} = \mathcal{K}_\sigma$ sets
- $l_0|_{\mathcal{B}or}$ = not strongly dominating sets

Themes in tree ideals investigation



- Consistency of $\omega_1 < add(t_0)$.
- When $\mathfrak{c} < cof(t_0)$?
- Consistency of $add(t_0) < cov(t_0)$.

When $\mathfrak{c} < cof(t_0)$?

Brendle, Khomskii, Wohofsky

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- $is(\mathbb{T}) = \mathfrak{c}$

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- Constant of 1-1 property holds in classical cases.
- $is(\mathbb{S}) = is(\mathbb{V}) = \mathfrak{c}$ in ZFC.
- $\mathfrak{b} \leq is(\mathbb{L})$ and $\mathfrak{d} \leq is(\mathbb{M})$

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- $add(fm_0|_{\mathcal{B}or}) \leq is(\mathbb{FM})$ but $add(fm_0|_{\mathcal{B}or}) = \omega_1$

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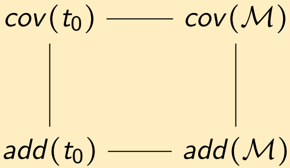
$min\{cov(fm_0|_{\mathcal{B}or}), \mathfrak{b}\} \leq is(\mathbb{FM})$ but $cov(\mathcal{M}) = cov(fm_0|_{\mathcal{B}or})$

Trees adding Cohen reals

Regarding $add(t_0) < cov(t_0)$

Theorem

If $\mathbb{T}$ adds Cohen reals with global reading then $\mathcal{M} \leq_T t_0$ and



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$$\begin{array}{ccc} cov(t_0) & \text{———} & cov(\mathcal{M}) \\ | & & | \\ add(t_0) & \text{———} & add(\mathcal{M}) \end{array}$$

Simply take $\Phi : \mathcal{M} \to t_0$, $\Phi(M) = \{x : \phi(x) \in M\}$

Examples:

- $\mathbb{FM}$
- $\mathbb{L}^2$ and $\mathbb{M}^3$
- $\mathbb{B}^2$

No tree inside

$$\mathcal{J}(\mathbb{T}) = \{B \in \mathcal{B}orel : \neg\exists T \in \mathbb{T}\ [T] \subseteq B\}$$

Is it equal $t_0|_{\mathcal{B}or}$? Is it even an ideal?

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  $(*)_{\mathbb{T}}$ holds for $\mathbb{B}^2$, $\mathbb{L}^2$, $\mathbb{M}^3$

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- $(*)_{\mathbb{T}} \to J(\mathbb{T})$ is not an ideal and $\mathbb{T}$ adds Cohen reals
  $(*)_{\mathbb{T}}$ holds for $\mathbb{B}^2$, $\mathbb{L}^2$, $\mathbb{M}^3$
- c.r.n. $\to J(\mathbb{T})$ is σ-ideal and $J(\mathbb{T}) = t_0|_{\mathcal{B}or}$
  $\mathbb{FM}$ has c.r.n.

The end

Thank you